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THE POINCARÉ SERIES OF A CODIMENSION FOUR GORENSTEIN RING IS RATIONAL

Carl JACOBSSON
Stockholm University, Mathematics Department, Box 6701, S-11385 Stockholm, Sweden

Andrew R. KUSTIN* and Matthew MILLER*
Mathematics Department, Stockholm University, Box 6701, S-11385 Stockholm, Sweden

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Dedicated to Jan-Erik Roos on his 50-th birthday

Let R be a regular local ring over a field k of characteristic different from two and let $S=R/I$ be a codimension four Gorenstein quotient with the same embedding dimension e as R. The Poincaré series of S is defined to be $P_S(z)=\sum \dim_k \operatorname{Tor}_i^S(k,k)z^i=\sum \dim_k \operatorname{Ext}_S^i(k,k)z^i$. We show that $P_S(z)=(1+z)^e/p(z)$, where $p(z)$ is a polynomial of one of four possible forms (explicitly given). As a corollary it follows that either S is a complete intersection or there exists a complete intersection $\tilde{R}=R/(a_1,a_2)$ with $a_i\in I$ such that $\tilde{R}\to S$ is a Golod homomorphism. The structure of the homotopy Lie algebra of S, $\pi^*(S)$, can then be elucidated: it is either finite-dimensional or the extension of the finite-dimensional Lie algebra $\pi^*(\tilde{R})$ by a free Lie algebra.

Introduction

Let R be a regular local ring over a field k not of characteristic two. Let $S=R/I$ be a codimension four Gorenstein quotient of R such that R and S have the same embedding dimension e_1. The Poincaré series of the local ring S is

$$P_S(z)=\sum_{i\geq 0} \dim_k \operatorname{Tor}_i^S(k,k)z^i=\sum_{i\geq 0} \dim_k \operatorname{Ext}_S^i(k,k)z^i.$$

After some preliminaries in Section 1, we compute this series in Section 2. We have the following result (Theorem 2.2, Corollary 2.3).

Theorem A. *Let S be as above. Then the Poincaré series of S is given by $P_S(z)=(1+z)^{e_1}/p(z)$, where $p(z)$ is a polynomial of one of four possible forms (explicitly given).*

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The Poincaré series is the generating series of the Yoneda Ext-algebra of $S, \mathrm{Ext}_S^*(k,k)$. This is a Hopf algebra and is the enveloping algebra of a graded Lie algebra $\pi^*(S)$, the homotopy Lie algebra of S; i.e. $U(\pi^*(S)) = \mathrm{Ext}_S^*(k,k)$. The dimension of $\pi^1(S)$ is e_1 and the dimension of $\pi^2(S)$ is the minimal number of generators of I. If S is a complete intersection, then $\pi^{\geq 3}(S) = 0$; otherwise $\pi^*(S)$ is infinite-dimensional. In Section 3 we use Theorem A to obtain the structure of $\pi^*(S)$, with the following results (Theorem 3.1, Corollary 3.2).

Theorem B. *Let S be as above. Then either S is a complete intersection or there exists a complete intersection $\tilde{R} = R/(a_1, a_2)$ with $a_i \in I$ (allowing the possibility $a_1 = a_2$) such that $\tilde{R} \twoheadrightarrow S$ is a Golod epimorphism.*

Corollary C. *Let S and $\tilde{R}$ be as in Theorem* B. *Then either $\pi^*(S)$ is finite-dimensional or $\pi^*(S)$ is the extension of the finite-dimensional Lie algebra $\pi^*(\tilde{R})$ by a free Lie algebra.*

Corollary D. *Let S be as above and M be a finitely generated S-module. Then the Poincaré series of M*

$$P_S^M(z) = \sum_{i \geq 0} \dim_k(\mathrm{Tor}_i^S(M,k))z^i$$

is a rational function.

If S is not a complete intersection, then by Corollary C the finitistic global dimension of $\mathrm{Ext}_S^*(k,k)$ is at most three and the λ-dimension of $\mathrm{Ext}_S^*(k,k)$ is one (see Roos [23]). Further results concerning the deviations of S are given in Section 3.

The main tool used to prove that $P_S(z)$ is rational is Avramov's theorem [3, Corollary 3.3], which converts the calculation of $P_S(z)$ to a calculation of the Poincaré series of $\Lambda = \mathrm{Tor}^R(S,k)$, provided the minimal resolution of S by free R-modules is a DG algebra. In [15, 17] Kustin and Miller proved that this hypothesis holds in the immediate cases of interest (all codimension four Gorenstein algebras, and certain higher codimension Gorenstein algebras). Just recently in [16] they have shown that in the codimension four case the algebra Λ has one of exactly four possible forms. The proof of Theorem B is then modeled on that of Jacobsson [13], who proved the same result in the case that S is a codimension three quotient of R.

1. Preliminary results

All rings and algebras are associative and have a unit element. Fix a field k of characteristic not two. A *local ring* is a commutative noetherian ring with a unique maximal ideal. A *local ring over* k is a local ring which has residue field k, and a map of local rings over k is a commutative triangle

$$\begin{tikzcd} R \arrow[rr] \arrow[dr] & & S \arrow[dl] \\ & k & \end{tikzcd}$$

The rest of the introductory material deals with graded algebras and modules; for more detail consult [9].

In this paper A will always denote a *graded-commutative algebra*. That is, $A=\bigoplus_{i=0}^{\infty}A_i$ with $a_ia_j=(-1)^{ij}a_ja_i\in A_{i+j}$ and $a_i^2=0$ if i is odd, for all $a_i\in A_i$ and $a_j\in A_j$. We shall assume A_0 is itself a local ring over k. Hence every graded-commutative algebra that we consider comes equipped with an augmentation homomorphism $\varepsilon: A \twoheadrightarrow k$; the augmentation ideal $I(A)$ is defined to be $\ker(\varepsilon)$ and always contains $A_+=\bigoplus_{i=1}^{\infty}A_i$. Sometimes we impose the additional hypotheses that A be *connected* (i.e. $A_0=k$) or *locally finite* (i.e. $\dim A_i<\infty$ for all i). If M is a graded A-module, then the *trivial extension* $A\ltimes M$ *of* A *by* M is the graded algebra with underlying graded A-module $A\oplus M$ and multiplication given by

$$(a_1,m_1)(a_2,m_2)=(a_1a_2,a_1m_2+(-1)^{\deg(a_2)\deg(m_1)}a_2m_1).$$

A *DG-algebra* is a graded-commutative algebra A with a differential $d:A_i\to A_{i-1}$ satisfying the Leibniz rule $d(a_ia_j)=(da_i)a_j+(-1)^ia_ida_j$ for $a_i\in A_i$ and $a_j\in A_j$. A *Γ-algebra* is a graded commutative algebra with divided powers: for each element a in A of positive even degree, there is an associated sequence of elements $\{a^{(0)},a^{(1)},a^{(2)},\dots\}$ satisfying $a^{(0)}=1$, $a^{(1)}=a$, $\deg a^{(k)}=k\deg a$, and a list of axioms, which we shall not need explicitly; see [9, Definition 1.7.1]. Many of the graded-commutative algebras that we consider are automatically Γ-algebras. For example, if $A=A_0$, or if A is an exterior algebra on a graded vector space all of whose elements have odd degree, or if A is the 'trivial algebra' $k[x_1,\dots,x_n]/(x_1,\dots,x_n)^2$, then A is a *Γ-algebra*. If the field of rational numbers is contained in A, then A is a Γ-algebra with $a^{(k)}=(1/k!)a^k$. Similarly if $A_i=0$ for $i>4$ and 2 is a unit in A, then A is a Γ-algebra. If V is a finite-dimensional graded vector space over k with a homogeneous basis $\{v_1,\dots,v_r,w_1,\dots,w_s\}$ where $\deg v_i$ are odd and $\deg w_j$ are even, then $E(V)$, the *exterior algebra on* V, is the Γ-algebra $\Lambda^{\cdot}(\bigoplus_{i=1}^{r}kv_i)\otimes k\langle w_1,\dots,w_s\rangle$, where $\Lambda^{\cdot}$ denotes the usual exterior algebra and $k\langle\cdots\rangle$ denotes the polynomial algebra with divided powers.

A DG-algebra with divided powers that also satisfies the condition $da^{(k)}=(da)a^{(k-1)}$ is called a *DGΓ-algebra*. If $\{x_i\}$ is a collection of homogeneous elements in a DGΓ-algebra X (usually cycles representing basis elements of $H(X)$), then the *divided polynomial algebra* $Y=X\langle\{S_i\};\ dS_i=x_i\rangle$ is a new DGΓ-algebra. As an X-module Y is free with basis $\{S_{i_1}^{(e_1)}\cdots S_{i_r}^{(e_r)}\}$; the grading is determined by setting $\deg S_i=1+\deg x_i$. The differential and multiplication are natural extensions of those on X. (The process is known as Tate's method of 'killing cycles'.) Gulliksen [6; 9, Proposition 1.9.3] has proved that if R,m,k is a local ring, then the Tate resolution $(X,d)=R\langle T_1,\dots\rangle$ of k is a minimal resolution in the sense that $dX \subseteq mX$. By its construction X is a DGΓ-algebra. Józefiak [14, Theorem 4.6] has extended Gulliksen's method to show that if A is a Γ-algebra, then the minimal homogeneous resolution X of k by free A-modules is a DGΓ-algebra. In this case $dX \subseteq I(A)X$.

We are ultimately concerned with the Poincaré series of a local ring S, m, k. If M is a finitely generated S-module, then the *Poincaré series of M as an S-module is*

$$P_S^M = \sum_{i=0}^{\infty} \dim_k \operatorname{Tor}_i^S(M,k) z^i \tag{1.1}$$

and the *Poincaré series of S* is $P_S = P_S^k$. In order to compute P_S we will often invoke Avramov's theorem (here Theorem 1.7) and calculate P_A, where A is a suitable chosen graded-commutative algebra. Defining the Poincaré series of a graded algebra is a little tricky, so at the risk of pedantry we shall spell out the details.

Let M and N be finitely generated graded modules over a graded-commutative algebra A. Then $\operatorname{Tor}_p^A(M,N)$ is a graded A-module with qth homogeneous piece $\operatorname{Tor}_{pq}^A(M,N)$. In other words if

$$X\colon \quad \cdots \to X_1 \to X_0 \to M \to 0$$

is a resolution of M by free A-modules, then each $X_p \otimes_A N = \sum_q (X_p \otimes_A N)_q$ is a graded A-module and

$$\operatorname{Tor}_{pq}^A(M,N) = \frac{\ker[(X_p \otimes N)_q \to (X_{p-1} \otimes N)_q]}{\operatorname{im}[(X_{p+1} \otimes N)_q \to (X_p \otimes N)_q]}\,.$$

Some authors (see for example Herzog and Steurich [11, 12]) consider a two-variable Poincaré series:

$$P_A^M(X,Y) = \sum_{p,q \ge 0} \dim_k \operatorname{Tor}_{pq}^A(M,k) X^p Y^q,$$

where $k \simeq A/I(A)$ via the augmentation. For us the Poincaré series of the graded module M over A is the condensation to a single-variable series:

$$P_A^M = \sum_{i=0}^{\infty} \left(\sum_{p+q=i} \dim_k \operatorname{Tor}_{pq}^A(M,k) \right) z^i = P_A^M(z,z). \tag{1.2}$$

If S is local and M is a finitely generated S-module, then P_S^M can be computed by (1.1), or equivalently by (1.2) if S and M are treated as graded objects concentrated in degree zero. Occasionally for typographic reasons we shall write $P(A,M)$ for P_A^M and $P(A)$ for P_A. We now offer a few examples of Poincaré series for graded algebras. These examples, and the following propositions, are results that we will need in the sequel to calculate the Poincaré series of a codimension four Gorenstein ring. They are, for the most part, analogs in the category of graded-commutative algebras for well-known results in the category of local algebras.

Example 1.1. (a) Let $A=E(\bigoplus_{i=1}^n kv_i)$ where $\deg(v_i)=1$ for $i=1,\dots,n$. Observe that $I(A)=A_+$. By [14, Theorem 5.2] the DGΓ-algebra $A\langle S_1,\dots,S_n; dS_i=v_i\rangle$ is a minimal resolution of k by free A-modules. Thus $\operatorname{Tor}^A(k,k)=A\langle S_1,\dots,S_n\rangle\otimes_A k$ is simply the homology of the complex

$$\cdots\xrightarrow{0} kS_1^{(2)}\oplus kS_1^{(1)}S_2^{(1)}\oplus\cdots\oplus kS_n^{(2)}\xrightarrow{0}\oplus kS_i^{(1)}\xrightarrow{0}k.$$

The symbol $S_i^{(1)}$ represents an element of $\operatorname{Tor}_{11}^A(k,k)$ and the symbol $S_i^{(1)}S_j^{(1)}$ represents an element of $\operatorname{Tor}_{22}^A(k,k)$. It is clear that $P_A^{-1}=(1-z^2)^n$. Notice that the coefficient of z^i in P_A is zero if i is odd. This can not happen for Poincaré series of local rings.

(b) If M is a graded A-module, then $M[-d]$ is the graded module with $M[-d]_i=M_{i-d}$. Since $\operatorname{Tor}_{pq}^A(M[-d],k)=\operatorname{Tor}_{pq-d}^A(M,k)$ we see that

$$P_A^{M[-d]}=z^dP_A^M. \tag{1.3}$$

The following result is due to Gulliksen [7, Theorem 2]; see also [3, Proposition 9.1].

Theorem 1.2. *If A is a graded-commutative, locally finite, connected k-algebra and M is a finitely generated A-module, then $P_{A\ltimes M}^{-1}=(1-zP_A^M)P_A^{-1}$.* □

Corollary 1.3. *If A is the graded-commutative algebra $k[X_1,\dots,X_n]/(X_1,\dots,X_n)^2$ with $\deg X_i=e_i\geq 0$, then $P_A^{-1}=1-z\sum_{i=1}^n z^{e_i}$.*

Proof. The proof is by induction on n. If $n=0$ the result is obvious. Let $B=k[X_1,\dots,X_{n+1}]/(X_1,\dots,X_{n+1})^2$ and suppose the result is true for $A=k[X_1,\dots,X_n]/(X_1,\dots,X_n)^2$. Let $e=e_{n+1}$ and $M=k[-e]$. We may identify M with the ideal (X_{n+1}) in B. Then $B=A\ltimes M$ and by Theorem 1.2 and formula (1.3) we have

$$\begin{aligned}P_B^{-1}&=(1-zP_A^M)P_A^{-1}=[1-z(z^eP_A)]P_A^{-1}=P_A^{-1}-z^{e+1}\\&=1-z\sum_{i+1}^{n+1}z^{e_i}.\end{aligned}\qquad\square$$

Notice that if $e_i=0$ for all i, then $P_A^{-1}=1-nz$, which agrees with the usual formula [5; 3, Lemma 6.6]

$$P_A=\frac{(1+z)^n}{1-\sum_{i=1}^n i\binom{n+1}{i+1}z^{i+1}}$$

for the Poincaré series of the ungraded trivial algebra $A=k[X_1,\dots,X_n]/(X_1,\dots,X_n)^2$.

The next result has a long history. Let R,m,k be a zero-dimensional Gorenstein local ring of embedding dimension greater than one. Let $\bar{R}=R/\mathrm{soc}(R)$, where $\mathrm{soc}(R)$ denotes $0:_R m$, the socle of R. Gulliksen [7] found that

$$P_{\bar{R}}^{-1}=P_R^{-1}-z^2 \tag{1.4}$$

if R is actually a complete intersection. Levin [19, Theorem 3.11] found that the same formula holds if R has embedding dimension three. Levin and Avramov [20] and, independently, Rahbar-Rochandel [22] showed that (1.4) holds without any extra hypotheses on R. In the graded case Avramov [3, Theorem 9.2] showed that if $A=E(\bigoplus_{i=1}^3 kv_i)$ with $\deg v_i=1$ for $i=1,2,3$ and $\bar{A}=A/A_3$, then $P_{\bar{A}}^{-1}=P_A^{-1}-z^{2+3}$. Herzog and Steurich [11] proved a similar result for certain cases in which A is a homology algebra. More generally, the graded analog of a zero-dimensional Gorenstein local ring is a Poincaré duality algebra. A connected, graded-commutative, locally finite k-algebra A (over a field of characteristic different from two) is called a *Poincaré duality algebra of length g* if $A_i=0$ for $i>g$, $A_g\simeq k$, and the pairings $A_i\times A_{g-i}\to A_g$ given by multiplication are perfect for all i. We thank Avramov for the proof of the following graded version of (1.4). A statement of this result in topological terms appears as Theorem 7.5.5 in Avramov's article in the collection Astérisque 113–114.

Theorem 1.4. *Let A be a Poincaré duality Γ-algebra over k of length g. If $\dim_k(A_+/A_+^2)\geq 2$ and $\bar{A}=A/A_g$, then $P_{\bar{A}}^{-1}=P_A^{-1}-z^{g+2}$.*

Proof. If every element of A has even degree, then A is actually a commutative Gorenstein ring and the proof is similar to the original proof [20, Theorem 2] in the ungraded case. Since we have no need for this case in this paper, we offer no more detail. Henceforth, we may assume that A has a non-zero element of odd degree.

Let $w\neq 0$ generate A_g and $t\neq 0$ be an element of A with the least possible odd degree. Since A is a Poincaré duality algebra, there is a homogeneous element u in A_+ with $ut=w$. We claim that $uA_+=wA=A_g$ and $u^2=0$. If there is a homogeneous element v in A_+ such that $uv\neq 0$ is not in A_g then by duality there is another homogeneous element v' in A_+ such that $uvv'=w$. Since $\deg v+\deg v'=\deg t$, one of $\deg v$, $\deg v'$ must be an odd integer strictly less than $\deg t$, contradicting the choice of t. If u has odd degree, then $u^2=0$ because A is a graded-commutative algebra. If u has even degree, then w has odd degree, and $\deg(u^2)\neq \deg w$. It follows from $uA_+=wA$ that $u^2=0$.

Consider the Tate resolution (X,d) of k over A. By its construction X is a DGΓ-algebra, and by Józefiak and Gulliksen [6; 14, Theorem 4.7] X is minimal in the sense that $dX\subset I(A)X=A_+X$. Select $x\in X_1$ such that $dx=t$ in $X_0=A$. Let $\bar{X}=X\otimes_A\bar{A}$. We claim that

$$\tilde{Z}(\bar{X})\subseteq u\bar{X}+d(\bar{X}). \tag{1.5}$$

(As usual Z denotes cycles of a complex, $\tilde{Z}_i=Z_i$ for $i\geq 1$, and $\tilde{Z}_0(\bar{X})=I(\bar{A})=d(\bar{X}_1)$.) Let x' represent a cycle $\bar{x}'$ in $\tilde{Z}_+(\bar{X})$; that is, x' is in X and $dx'=wy$ for some y in X. Then

$$dx' = wy = uty = u(dx)y = d(uxy) \pm uxdy$$

and since dy is in A_+X and $uA_+ = wA$, we see that $\pm uxdy = xwy_1$ for some y_1 in X. The exact same reasoning shows that $xwy_1 = xuty_1 = x(dx)uy_1 = d(ux^{(2)}y_1) \pm x^{(2)}udy_1$ since X is a DGΓ-algebra. Continuing in this manner, we obtain

$$dx' = d(uxy + ux^{(2)}y_1 + \cdots + ux^{(i)}y_{i-1}),$$

the sum terminating when $i(\deg x) \geq \deg x'$. Since X is acyclic x' is in $uX + d(X)$, which establishes (1.5).

We may now apply [20, Lemma 2.1], which can be proved in the graded setting exactly as in the original ungraded case. The result is:

Lemma. *Let A be a connected, graded-commutative k-algebra and U a* DGΓ-*algebra such that*

(a) *each U_i is a free A-module and $U_0 = A$,*

(b) *$dU \subseteq A_+U$ and $dU_1 = A_+$,*

(c) *there is a graded submodule $M \subseteq A_+U$ such that $M^2 = 0$ and $\tilde{Z}(U) \subseteq M + dU$.*

Then the minimal resolution of k has the form $Y = U \otimes_A T(F)$ where F is a free A-module satisfying $F \otimes_A k \cong \tilde{H}(U)[-1]$ and $T(F)$ is the tensor algebra of F over S. $\square$

We apply the lemma with $U = \bar{X}$ and $M = u\bar{X}$, concluding that $\bar{X} \otimes_{\bar{A}} T(F)$ is a minimal resolution of k by free $\bar{A}$-modules, where

$$(F_p \otimes_{\bar{A}} k)_q \cong \tilde{H}_{p-1}(\bar{X})_q \cong \operatorname{Tor}^A_{p-1\,q}(\bar{A}, k) \quad \text{for all } p \geq 2 \text{ and } q \geq 0.$$

It is immediate that

$$P_{\bar{A}} = P_A(1 - z(P_A^{\bar{A}} - 1))^{-1}. \tag{1.6}$$

Using the exact sequence

$$0 \to k[-g] \cong wA \to A \to \bar{A} \to 0$$

together with $\operatorname{Tor}^A_{pq}(\bar{A}, k) \cong \operatorname{Tor}^A_{p-1\,q}(wA, k)$ for $p \geq 1$ and $q \geq 0$, and formula (1.3), we obtain $P_A^{\bar{A}} - 1 = z^{g+1}P_A$. Substitution into (1.6) yields the desired result. $\square$

The key step in our calculation of a Poincaré series of a local ring is Avramov's theorem, which shifts most of the work to calculating the Poincaré series of a Tor-algebra, which often has a much simpler structure. To state his theorem we must introduce the notion of small homomorphism. Many equivalent definitions can be found in [3, Theorem 3.1].

Definition 1.5. A homomorphism $f: R \to S$ of local rings over k is *small* if the induced map

$$f_* = \mathrm{Tor}^f(k,k) : \mathrm{Tor}^R(k,k) \to \mathrm{Tor}^S(k,k)$$

is injective. An ideal I in R is *small* if $R \to R/I$ is a small homomorphism.

Example 1.6. (a) If R, m, k is a regular local ring over k and I is an ideal contained in m^2, then I is small; see [3, Example 3.11]. The underlying reason is that the minimal (Tate) resolution of k over $S = R/I$ contains the Koszul complex $\mathbb{K}^S$ as an S-module summand and $\mathbb{K}^S = \mathbb{K}^R \otimes_R S$ since $I \subseteq m^2$, where $\mathbb{K}^R$ is the Koszul resolution of k over R.

(b) Let X denote a collection of indeterminates $\{X_1, \dots, X_n\}$. An ideal I in $\mathbb{Z}[X]$ is *generically perfect of grade* g if $g = \mathrm{grade}\, IR[X] = \mathrm{pd}_{R[X]} R[X]/IR[X]$ for R equal to $\mathbb{Z}$ or $\mathbb{Z}/p\mathbb{Z}$ for some prime p. (The *grade* of an ideal J in a ring R is the length of the longest R-sequence contained in J.) Let I be a generically perfect ideal of $\mathbb{Z}[X]$ contained in $(X_1, \dots, X_n)^2$ and suppose ϱ is a map of rings from $\mathbb{Z}[X]$ to a local ring R, m with $\varrho(X_i)$ in m for all i. If $\mathrm{grade}(IR) = \mathrm{grade}\, I$, then IR is a small ideal in R by [3, Theorem 6.2].

We can now state Avramov's theorem.

Theorem 1.7. [3, Corollary 3.3]. *Let $f : R \to S$ be a small homomorphism of local rings over k. If f gives S the structure of a finitely generated R-module and the minimal R-free resolution of S has the structure of a DG-algebra, then $P_R P_S^{-1} = P_\Lambda^{-1}$, where $\Lambda = \mathrm{Tor}^R(S, k)$.* □

By Kustin and Miller [15] the hypothesis that the minimal R-free resolution of S be a DG-algebra is satisfied if R is a Gorenstein local ring in which 2 is a unit and $S = R/I$ for I a grade four Gorenstein ideal. It also holds if I is a codimension g 'Herzog ideal', as explained in [17] and Proposition 2.4.

2. Poincaré series

If I is a grade four Gorenstein ideal in a regular local ring R over k and char $k \neq 2$, then we shall prove (Corollary 2.3) that $P_R P_{R/I}^{-1}$ is a polynomial. The following result of Kustin and Miller [16] is used to compute P_Λ, as required in Theorem 1.7.

Theorem 2.1. *Let R, m, k be a Gorenstein local ring in which 2 is a unit, and assume k has square roots. Let I be a grade four Gorenstein ideal in R and $\Lambda = \mathrm{Tor}^R(R/I, k)$. Then Λ is a Poincaré duality Γ-algebra. Moreover there are bases $\{x_1, \dots, x_n\}$ for Λ_1, $\{y_1, \dots, y_{n-1}, y_1', \dots, y_{n-1}'\}$ for Λ_2, $\{z_1, \dots, z_n\}$ for Λ_3, and $\{w\}$ for Λ_4 so that the multiplication $\Lambda_i \times \Lambda_{4-i} \to \Lambda_4 = k$ is given by $x_i z_j = \delta_{ij} w$, $y_i y_j' = \delta_{ij} w$, $y_i y_j = 0 = y_i' y_j'$, and the other products in Λ are given by one of the following cases:*

(A) *The ideal I is generated by a regular sequence in which case $\Lambda = E(\bigoplus_{i=1}^4 kx_i)$.*

(B) *All products in $\Lambda_1\Lambda_1$ and $\Lambda_1\Lambda_2$ are zero,*

(c) *All products in $\Lambda_1\Lambda_1$ and $\Lambda_1\Lambda_2$ are zero except those indicated in the multiplication tables:*

	x_1	x_2	x_3
x_1	0	y_3	$-y_2$
x_2	$-y_3$	0	y_1
x_3	y_2	$-y_1$	0

	y_1'	y_2'	y_3'
x_1	0	z_3	$-z_2$
x_2	$-z_3$	0	z_1
x_3	z_2	$-z_1$	0

(D) *There is an integer p such that $x_{p+1}x_i = y_i$, $x_iy_i' = z_{p+1}$, and $x_{p+1}y_i' = -z_i$ for $1 \le i \le p$, and all other products in $\Lambda_1\Lambda_1$ and $\Lambda_1\Lambda_2$ are zero.* □

We assume for the remainder of this paper that the residue field k contains square roots. (Since the Poincaré series P_R and $P_{R/I}$ are unchanged under flat extension this condition can be obtained.) Let $\mu(I)$ denote the minimal number of generators of I.

Theorem 2.2. *Let R be a Gorenstein local ring over a field k with* char $k \neq 2$, *and let $\Lambda = \mathrm{Tor}^R(S,k)$, where I is a grade four Gorenstein ideal in R with $\mu(I) = n$ and $S = R/I$. Then Λ has one of the forms* (A), (B), (C), (D) *as described above and P_Λ^{-1} is respectively equal to*

(A) $\quad (1-z^2)^4,$

(B) $\quad 1 - nz^2 - 2(n-1)z^3 - nz^4 + z^6,$

(C) $\quad 1 - nz^2 - (2n-5)z^3 - (n-6)z^4 + 2z^5 - z^6 - z^7,$

(D) $\quad 1 - nz^2 - (2n-2-p)z^3 + (2p+1-n)z^4 + (p+1)z^5 - z^7.$

If, further, I is small, then $P_R P_S^{-1} = P_\Lambda^{-1}$.

Proof. The last statement is just Theorem 1.7. If $\Lambda = E(\bigoplus_{i=1}^4 kx_i)$, then by Example 1.1(a), $P_\Lambda^{-1} = (1-z^2)^4$. If Λ has form (B), then Avramov [3, Proposition 9.6] has shown that $P_\Lambda^{-1} = 1 + z - zP_R^S + z^5 + z^6$. Reading off the Betti numbers in a minimal R-free resolution of S, we obtain $P_R^S = 1 + nz + 2(n-1)z^2 + nz^3 + z^4$ and formula (B) follows.

If Λ has form (C) we shall compute $P_{\bar\Lambda}$ and then apply Theorem 1.4 to the Poincaré duality Γ-algebra Λ. The following decomposition of $\bar\Lambda = \Lambda/\Lambda_4$ was suggested to us by Avramov. Let $A = E(\bigoplus^3{}_{i=1} kx_i)$ and $\bar A = A/A_3$. Set

$$W = k[-1]^{n-3} \oplus k[-2]^{2n-8} \oplus k[-3]^{n-3}, \quad V = \bar A_+[-1], \text{ and } B = \bar A \ltimes V.$$

It is not difficult to see that $\bar\Lambda = B \ltimes W$. By Theorems 1.4 and 1.2 we have

$$P_\Lambda^{-1} = z^6 + P_{\bar\Lambda}^{-1} = z^6 + (1 - zP_B^W)P_B^{-1}.$$

By formula (1.3) $P_B^W = QP_B$ where $Q=(n-3)z+(2n-8)z^2+(n-3)z^3$. Application of Theorem 1.2 yields

$$P_A^{-1}=P_B^{-1}+z^6-zQ=(1-zP_{\bar{A}}^V)P_{\bar{A}}^{-1}+z^6-zQ.$$

Now $V=\bar{A}_+[-1]$ and $\bar{A}_+$ is the kernel of the augmentation $\bar{A}\to k$, so by (1.3) and dimension shifting with $\mathrm{Tor}^{\bar{A}}(\cdot,k)$ we obtain $P_{\bar{A}}^V = z(P_{\bar{A}}-1)/z$. Hence

$$P_A^{-1}=(1+z)P_{\bar{A}}^{-1}-z+z^6-zQ.$$

Finally we apply Theorem 1.4 and Example 1.1(a) to see that $P_{\bar{A}}^{-1}=P_{\Lambda}^{-1}-z^5=(1-z^2)^3-z^5$; formula (C) follows directly.

Now suppose A has form (D) with $\dim A_1A_1=p$. Let Λ be the subalgebra $k[x_1,\dots,x_p,y_1',\dots,y_p',z_{p+1}]$. Clearly Λ is a Poincaré duality algebra of length three in which all products of generators are zero except for those that give the pairing, namely $x_iy_i'=z_{p+1}$. Let $\bar{A}=A/A_4$ and $\bar{\Lambda}=\Lambda/\Lambda_3$. Evidently $\bar{\Lambda}$ is a trivial algebra, and if we let $C=k[x_{p+1}]/(x_{p+1})^2$ then $B=\Lambda\otimes_k C$ is also a Poincaré duality Γ-algebra of length four, where $B_4=k(z_{p+1}\otimes x_{p+1})$. Since k is a field the minimal B-free resolution of k is simply obtained by taking the tensor product of the respective minimal free resolutions over Λ and C (use the standard Künneth formula); thus $P_B=P_\Lambda P_C$. Let $\bar{B}=B/B_4$ and M be the trivial $\bar{B}$-module

$$k[-1]^{n-p-1}\oplus k[-2]^{2n-2-2p}\oplus k[-3]^{n-p-1}.$$

The key observation is that $\bar{A}=\bar{B}\ltimes M$, which the reader may easily verify. Then by Theorems 1.4 and 1.2 we have

$$P_A^{-1}=z^6+P_{\bar{A}}^{-1}=z^6+\left(1-zP_{\bar{B}}^M\right)P_{\bar{B}}^{-1}$$

and $P_{\bar{B}}^M=QP_{\bar{B}}$ where $Q=(n-p-1)z+(2n-2-2p)z^2+(n-p-1)z^3$ by using (1.3). Two applications of Theorem 1.4 yield

$$P_A^{-1}=z^6+P_{\bar{B}}^{-1}-zQ=P_B^{-1}-zQ=(P_{\bar{\Lambda}}^{-1}+z^5)P_C^{-1}-zQ.$$

By Corollary 1.3 we have $P_{\bar{\Lambda}}^{-1}=1-pz^2-pz^3$ and $P_C^{-1}=1-z^2$, and formula (D) for P_A^{-1} follows readily. □

To relate $P_R P_{R/I}^{-1}$ to P_A^{-1} we need I to be small. For R a regular local ring smallness just amounts to $I\subseteq m^2$. Should this condition fail, we have, in effect, an ideal in lower codimension with well-known Poincaré series.

Corollary 2.3. *Let R,m,k be a regular local ring over a field k of characteristic different from two. Let I be a grade four Gorenstein ideal in R with $\mu(I)=n$. If I is a complete intersection, then $P_R(P_{R/I})^{-1}=(1+z)^s(1-z^2)^{4-s}$ where $s=\dim_k(I+m^2)/m^2$. If I is not a complete intersection and is not contained in m^2, then*

$$P_R(P_{R/I})^{-1}=1+z-(n-1)z^2-2(n-1)z^3-(n-1)z^4+z^5+z^6.$$

If I is contained in m^2, then $P_R(P_{R/I})^{-1}$ is one of the polynomials listed in Theorem 2.2.

Proof. The first assertion is classical and can be found in [3, Proposition 4.2] or [9, Corollary 3.4.3]; or one can specialize to the small case and use Example 1.1(a). For the second situation choose x in $I \setminus m^2$ and let $\bar{I}$ be the image of I in the regular local ring $\bar{R} = R/(x)$. Now $\bar{R}/\bar{I} \cong R/I$, so $\bar{I}$ is a Gorenstein ideal of grade three which is not a complete intersection. By Avramov [3, Theorem 8.2] or Wiebe [24, Satz.9] we know that

$$P_{\bar{R}}(P_{\bar{R}/\bar{I}})^{-1} = 1 - (n-1)z^2 - (n-1)z^3 + z^5.$$

The result follows since $P_R = (1+z)P_{\bar{R}}$. If, finally, $I \subseteq m^2$, then I is small and Theorem 2.2 applies. $\square$

As a further application of these techniques we calculate the Poincaré series of an algebra R/I defined by a grade g 'Herzog ideal'. Herzog and Steurich [12] have already made this calculation, but they did not know that the minimal resolution of R/I by free R-modules admits the structure of a DG-algebra. Consequently they were obliged to make some rather nasty computations of Massey products. Our proof is in essence the same, but the ugly details are masked by Avramov's theorem.

Proposition 2.4. *Let R, m, k be a local ring over k and let $v, a_1, \ldots, a_g$, and x_{ij} for $1 \le i \le g$, $1 \le j \le g-1$ be elements of m. Let c_i be $(-1)^{i+1}$ times the determinant of the submatrix of X formed by deleting row i. Let I be the ideal generated by $\sum a_i x_{ij}$ for $1 \le j \le g-1$ and $c_i + va_i$ for $1 \le i \le g$. If grade $I = g$, then I is small and*

$$P_R(P_{R/I})^{-1} = (1+z)^g[(1-z)^{g-1} - z].$$

Proof. The ideal I is generically perfect by [10, Corollary 4.5 and Example 3] or [17], so by Example 1.6(b) the ideal I is small. An explicit DG-algebra structure on the minimal R-free resolution $\mathbb{F}$ of R/I can be found in [17]. By Avramov's theorem $P_R(P_{R/I})^{-1} = P_A^{-1}$ for $A = \operatorname{Tor}^R(R/I, k) = \mathbb{F} \otimes_R k$. Let $U = \bigoplus_{i=1}^{g-1} kz_i$ and $V = \bigoplus_{i=1}^{g} kx_i$ be vector spaces with basis elements z_i and x_i all of degree 1. From [17] we have

$$A_i = \begin{cases} E_0(U) & \text{if } i = 0, \\ E_1(U) \oplus E_1(V) & \text{if } i = 1, \\ E_i(U) \oplus E_i(V) \oplus E_{i-1}(U) & \text{if } 2 \le i \le g-2, \\ E_{g-1}(V) \oplus E_{g-2}(U) & \text{if } i = g-1, \\ E_{g-1}(U) & \text{if } i = g, \end{cases}$$

and the multiplication $A_i \times A_j \to A_{i+j}$ is given by

$$\begin{bmatrix} \lambda_i \\ \mu_i \\ \lambda_{i-1} \end{bmatrix} \begin{bmatrix} \lambda_j \\ \mu_j \\ \lambda_{j-1} \end{bmatrix} = \begin{bmatrix} (1 - \delta_{i+j,g-1})\lambda_i \wedge \lambda_j \\ \delta_{i0}\lambda_i\mu_j + \delta_{j0}\lambda_j\mu_i \\ \lambda_i \wedge \lambda_{j-1} + (-1)^j \lambda_{i-1} \wedge \lambda_j - \delta_{i+j,g}\mu_i \wedge \mu_j \end{bmatrix}.$$

It is easy to see that $\Lambda_i \times \Lambda_{g-i} \to \Lambda_g = k$ is a perfect pairing for all i. Let $E = E(U)$, $\bar{E} = E/E_{g-1}$, $M = \bar{E}_+[-1]$, and W be the graded vector space $\bigoplus_{i=1}^{g-1} E_i(V)$. There is no difficulty verifying that $\bar{\Lambda} = (\bar{E} \ltimes M) \ltimes W$, where $\bar{E} \ltimes M$ acts trivially on W. In particular the multiplication $(\bar{E} \ltimes M)_i \times (\bar{E} \ltimes M)_j \to (\bar{E} \ltimes M)_{i+j}$ given by

$$\begin{bmatrix} \lambda_i \\ 0 \\ \lambda_{i-1} \end{bmatrix} \begin{bmatrix} \lambda_j \\ 0 \\ \lambda_{j-1} \end{bmatrix} = \begin{bmatrix} \lambda_i \wedge \lambda_j \\ 0 \\ \lambda_i \wedge \lambda_{j-1} + (-1)^{ij} \lambda_j \wedge \lambda_{i-1} \end{bmatrix}$$

agrees with the formula given above since $(-1)^j \lambda_{i-1} \wedge \lambda_j = (-1)^{ij} \lambda_j \wedge \lambda_{i-1}$. We proceed as in part (C) of the proof of Theorem 2.2 (taking $A = \bar{E} \ltimes M$):

$$\begin{aligned} P_{\Lambda}^{-1} &= z^{g+2} + P_{\bar{\Lambda}}^{-1} = z^{g+2} + (1 - zP_A^W) P_A^{-1} \\ &= z^{g+2} + \left(1 - z \sum_{i=1}^{g-1} \binom{g}{i} z^i P_A\right) P_A^{-1} \\ &= z^{g+2} - z((1+z)^g - 1 - z^g) + P_A^{-1}, \\ P_A^{-1} &= (1 - zP_{\bar{E}}^M) P_{\bar{E}}^{-1} = (1 - z(P_{\bar{E}} - 1)) P_{\bar{E}}^{-1} \\ &= -z + (1+z) P_{\bar{E}}^{-1} \\ &= -z + (1+z)(P_E^{-1} - z^{g+1}). \end{aligned}$$

The result follows since $P_E^{-1} = (1 - z^2)^{g-1}$. □

3. Golod homomorphisms

It is well-known [5, 19, 23] that if $f: R \to S$ is a surjective map of local rings over k, then there is a coefficient-wise inequality of Poincaré series

$$P_S \le P_R (1 - z(P_R^S - 1))^{-1}.$$

If equality holds we say f if a *Golod map*. Our main result in this section is that every codimension four Gorenstein algebra which is a quotient of a regular local ring can be reached from some regular local ring by a sequence of Golod maps. Equivalent definitions and various related conditions are discussed in Avramov [2, 3], Levin [18, 19], and Löfwall [21]. In particular, if $\pi^*(S)$ is the graded Lie algebra associated to S (the homotopy Lie algebra of S, denoted $\pi^*(S)$, is determined by the property that its universal enveloping algebra $U(\pi^*(S))$ is isomorphic to $\mathrm{Ext}_S(k, k)$), then f is a Golod map if and only if the sequence

$$0 \to L(W) \to \pi^*(S) \xrightarrow{f^*} \pi^*(R) \to 0$$

is exact, where $L(W)$ is the free graded Lie algebra on a vector space basis of $W = \mathrm{Tor}_+^R(S, k)[-1]$. For the interesting case when S is not a complete intersection, Avramov [1] showed that if S is a codimension four Gorenstein algebra, then $\pi^*(S)$

contains a free Lie algebra on two generators. He used the then available primitive knowledge of the multiplicative structure of Λ. We use [16] to give a description of $\pi^*(S)$ as an extension of a finite Lie algebra $\pi^*(\tilde{R})$ by a free Lie algebra $L(W)$, thus determining the graded vector space structure and most of the Lie algebra structure of $\pi^*(S)$.

One of the key examples of a Golod map, found already in [5, Theorem 3.7], motivated Levin to introduce the concept. If R is an arbitrary local ring and $b \in m^2$ is not a zero-divisor, then $R \to R/(b)$ is a Golod homomorphism. It follows by induction that if $\mathbf{b} = b_1, \dots, b_g$ is a regular sequence contained in m^2, then $P_R(P_{R/(\mathbf{b})})^{-1} = (1-z^2)^g$, as in Corollary 2.3. Roos [23] denotes by $\mathcal{AG}$ the class of rings that can be reached from a regular local ring by a finite sequence of Golod surjections; we call such rings *Golod attached.* We recall two of the many properties of rings in $\mathcal{AG}$.

The qth deviation, $e_q(S)$, of a local ring (S,k) is the number of variables of degree q adjoined in a minimal Tate resolution of k over S, and is also $\dim_k \pi^q(S)$. (The classical deviations, as in [9], are $\varepsilon_q = e_{q+1}$ for $q \geq 0$.) It is an open question ([9, p. 154] or [1, Conjecture C_3]) whether $e_q(S) > 0$ for all $q \geq 1$ if S is not a complete intersection. Gulliksen [8] further conjectured that $e_q(S) < e_{2q}(S)$ for all odd $q > 1$. Avramov and Halperin [4] have proved the first conjecture for Gorenstein rings of codimension four; Gulliksen [8] has proved his own conjecture for Gorenstein rings of codimension three. Recently, Jacobsson [13, Corollary 1] has proved Gulliksen's conjecture for all rings in the class $\mathcal{AG}$; hence we shall see that it also holds for Gorenstein rings of codimension four.

Roos [23, Theorem 5] has shown that if S is attached to a regular local ring R,k by a sequence of s Golod surjections, then the finitistic global dimension of the Hopf algebra $\mathrm{Ext}_S(k,k)$ is at most s. Hence we shall see that fin.gl.dim. $\mathrm{Ext}_S(k,k) \leq 3$ if S is a codimension four Gorenstein ring that is not a complete intersection (and is equal to four if S is a complete intersection of codimension four). All of these results follow from our main theorem:

Theorem 3.1. *If R, m, k is a regular local ring over a field k of characteristic not two and I is a grade four Gorenstein ideal in R, then there is a regular sequence* $\mathbf{a}$ *in I so that the natural map $R/(\mathbf{a}) \to R/I$ is a Golod map.*

Proof. (Recall that we are assuming k has square roots; see Theorem 2.2.) The proof is broken down according to the cases of Corollary 2.3. If I is generated by a regular sequence, then the identity map $R/I \to R/I$ is the desired Golod map. If I is not a complete intersection and is not contained in m^2, then let a_0 be an element in $I \setminus m^2$ and let $\bar{I}$ be the image of I in the regular local ring $\bar{R} = R/(a_0)$. Then $R/I \cong \bar{R}/\bar{I}$ and $\bar{I}$ is a grade three Gorenstein ideal which is not a complete intersection. By Jacobsson [13, Appendix] there is an element a_1 in R such that $\bar{a}_1$ is not a zero-divisor in $\bar{R}$ and $\bar{R}/(\bar{a}_1) \to \bar{R}/\bar{I}$ is Golod. Then a_0, a_1 is a regular sequence and $R/(a_0, a_1) \to R/I$ is Golod. If I is not a complete intersection, but is contained in m^2, then I is a small and $\Lambda = \mathrm{Tor}^R(R/I, k)$ has one of the forms (B), (C), (D) described in Theorem 2.2.

Case (B). We shall prove that there is a regular element a in I such that $R/(a)\to R/I$ is Golod. We let $(\hat{Y},d)$ be the Koszul resolution of k over R and we set $Y=R/I\otimes_R \hat{Y}$. Then $\Lambda=H(Y)$ and a basis for the reduced homology is given by the equivalence classes of the cycles $x_1,\dots,x_n$ in degree 1; $y_1,\dots,y_{n-1},y_1',\dots,y_{n-1}'$ in degree 2; $z_1,\dots,z_n$ in degree 3; and w in degree 4. After possibly modifying x_1 by a boundary, we may assume that there is a pre-image $\hat{x}_1$ of x_1 in $\hat{Y}$ such that $d\hat{x}_1=a$ is a regular element in R (and obviously $a\in I$ since x_1 is a cycle in Y). In Y all products of the listed cycles are boundaries except that $x_iz_j=\delta_{ij}w$ and $y_iy_j'=\delta_{ij}w$. Let $\tilde{R}=R/(a)$ and $\tilde{Y}=\hat{Y}\otimes_R\tilde{R}$. The image $\tilde{x}_1$ of $\hat{x}_1$ in $\tilde{Y}$ is a cycle; if we adjoin a variable S of degree two so that $dS=\tilde{x}_1$, then $\tilde{Y}\langle S\rangle$ is a minimal $\tilde{R}$-resolution of k.

To prove that $\tilde{R}\to R/I$ is Golod it suffices to prove that $P_{\tilde{R}}(P_{R/I})^{-1}=1-z(P^{R/I}-1)$, or equivalently,

$$P_{\tilde{R}}^{R/I}=1-\frac{1}{z}\left(\frac{P_{\tilde{R}}}{P_R}\frac{P_R}{P_{R/I}}-1\right). \tag{3.2}$$

Now a is a regular element in m^2, so $R\to\tilde{R}$ is Golod and $P_{\tilde{R}}P_R^{-1}=(1-z^2)^{-1}$. Using Theorem 2.2 to rewrite $P_RP_{R/I}^{-1}$ we find that it suffices to prove

$$P_{\tilde{R}}^{R/I}=1+(n-1)z+(2n-2)z^2+(2n-1)z^3+(2n-2)\sum_{i=4}^{\infty}z^i \tag{3.3}$$

where $n=\mu(I)$.

The Poincaré series $P_{\tilde{R}}^{R/I}$ is obtained by studying the homology of $\tilde{Y}\langle S\rangle\otimes_{\tilde{R}}R/I=Y\langle S\rangle$. For each $i>0$ we select and fix cycles of $Y\langle S\rangle$ with the form described below:

$$\begin{cases} x_jS^{(i)}+\text{lower order terms in } S, & 2\le j\le n,\\ z_jS^{(i-1)}+\cdots, & 2\le j\le n,\end{cases} \tag{3.4}$$

$$\begin{cases} y_jS^{(i-1)}+\cdots, & 1\le j\le n-1,\\ y_j'S^{(i-1)}+\cdots, & 1\le j\le n-1.\end{cases} \tag{3.5}$$

It is not difficult to see that such cycles exist, and it is clear that

$$\{1\},\qquad \{x_j\mid 2\le j\le n\},\qquad \{y_j,y_j'\mid 1\le j\le n-1\},$$
$$\{z_i\mid 1\le i\le n\}\cup\{x_jS^{(1)}\mid 2\le j\le n\}$$

represent bases of $H_i(Y\langle S\rangle)$ for $i=0,1,2,3$. For $i\ge 2$ we will show that the $2n-2$ cycles in (3.4) represent a basis of $H_{2i+1}(Y\langle S\rangle)$ and the $2n-2$ cycles in (3.5) represent a basis of $H_{2i}(Y\langle S\rangle)$. Once this has been accomplished, (3.3) follows immediately.

First we demonstrate that we have a generating set for $H(Y\langle S\rangle)$; we induct on the divided power degree in S. We shall write l.o.t.(S) to denote 'lower order terms in S'. If $\eta=\xi S^{(i)}+\text{l.o.t.}(S)$ is a cycle in $Y\langle S\rangle$, then ξ is a cycle in Y. Recall that $dS^{(i+1)}=x_1S^{(i)}$ and $d(z_1S^{(i+1)})=wS^{(i)}$. It is not difficult to select χ, an R-linear combination of cycles from (3.4) and (3.5), and an element ϕ of $(1,z_1)S^{(i+1)}$ so that the leading term of $\eta-[\chi+d\phi]$ is $(u+u'z_1+db)S^{(i)}$ for some $b\in Y$ and $u,u'\in R$.

Then

$$\begin{aligned}0 &= d(\eta - [\chi + d(\phi + bS^{(i)})])\\ &= (ux_1 - u'z_1x_1)S^{(i-1)} + (db')S^{(i-1)} + \text{l.o.t.}(S)\\ &= (ux_1 + u'w + db')S^{(i-1)} + \text{l.o.t.}(S)\end{aligned}$$

for some $b' \in Y$. Since $i \geq 2$ it follows that the images $\bar{u}$ and $\bar{u}'$ of u and u' respectively are zero in R/m and $\eta - \chi$ differs by a boundary from a cycle that has degree less than i in S. By induction this cycle differs by a boundary from a linear combination of cycles from (3.4) and (3.5); thus so does η.

To conclude the proof we show that the cycles of (3.4) and (3.5) represent linearly independent classes in homology. Suppose that some R-linear combination of these cycles is a boundary. Observe that if $d(\sum s_j S^{(j)}) = \xi S^{(i)} + \text{l.o.t.}(S)$ is a homogeneous boundary element in $Y\langle S\rangle$, then (noting that $\deg s_{i+1} = \deg s_i - 2$)

$$\begin{aligned}\xi &= ds_i + (-1)^{\deg s_i} s_{i+1}x_1,\\ 0 &= ds_{i+1} + (-1)^{\deg s_i} s_{i+2}x_1.\end{aligned} \tag{3.6}$$

By degree considerations there are just three cases to handle:

(a) If $(\sum_{l=2}^{n} \alpha_l x_l)S^{(i)} + \text{l.o.t.}(S)$ is a boundary, then $\sum_{l=2}^{n} \alpha_l x_l = ds_i + s_{i+1}x_1$ and s_{i+1} must be in R/I. Since $\{x_1, \ldots, x_n\}$ is a basis of $H_1(Y)$, all $\bar{\alpha}_l$ must be zero.

(b) If $(\sum_{l=1}^{n-1} \alpha_l y_l + \alpha_l' y_l')S^{(i)} + \text{l.o.t.}(S)$ is a boundary, then by (3.6) $\deg s_{i+2} = -1$; hence s_{i+1} is a cycle of degree 1 and since A has form (B) the product $s_{i+1}x_1$ is a boundary. It follows that $\sum \alpha_l y_l + \alpha_l' y_l'$ is a boundary, and therefore $\bar{\alpha}_l = \bar{\alpha}_l' = 0$ for all l.

(c) If $(\sum_{l=2}^{n} \alpha_l z_l)S^{(i)} + \text{l.o.t.}(S)$ is a boundary, then by (3.6), $\deg s_{i+2} = 0$ and s_{i+2} is in R/I. Since x_1 is not a boundary in Y we conclude that s_{i+2} is in m. Thus $s_{i+2} = dt$ for some t in Y_1, and $y = s_{i+1} + tx_1$ is a cycle in Y_2. Now we have $\sum \alpha_l z_l = ds_i + (y - tx_1)x_1$. Since Y is a DG-algebra, $x_1^2 = 0$; and since A has form (B), yx_1 is a boundary. Hence $\sum \alpha_l z_l$ is a boundary and all $\bar{\alpha}_l$ are zero.

Case (C). We shall prove that there is a regular sequence a_1, a_2 in I so that the natural map $R/(a_1, a_2) \to R/I$ is Golod. Both $\hat{Y}$ and Y are defined as in case (B) above. The multiplication on Y is given by $x_i z_j = \delta_{ij} w$, $y_i y_j' = \delta_{ij} w$, and with the exception of the products listed below, all other products of cycles are boundaries. Since we are lifting the multiplication from $A = H(Y)$ there is some harmless indeterminancy which we denote '+b', i.e. 'plus a boundary'. Note, however, that since Y is a DG-algebra $x_i^2 = 0$ exactly.

	x_1	x_2	x_3
x_1	0	$y_3 + \text{b}$	$-y_2 + \text{b}$
x_2	$-y_3 + \text{b}$	0	$y_1 + \text{b}$
x_3	$y_2 + \text{b}$	$-y_1 + \text{b}$	0

	y_1'	y_2'	y_3'
x_1	b	$z_3 + \text{b}$	$-z_2 + \text{b}$
x_2	$-z_3 + \text{b}$	b	$z_1 + \text{b}$
x_3	$z_2 + \text{b}$	$-z_1 + \text{b}$	b

For each $i = 1, 2$, let $\hat{x}_i$ be a preimage of x_i in $\hat{Y}$ and let $d\hat{x}_i = a_i$. By modifying $\hat{x}_1$

and $\hat{x}_2$ by boundaries, if necessary, we may assume that a_1, a_2 is a regular sequence contained in I. Let $\tilde{R}=R/(a_1,a_2)$ and $\tilde{Y}=\hat{Y}\otimes_R \tilde{R}$. We adjoin variables S_1 and S_2 of degree two to kill $\tilde{x}_1$ and $\tilde{x}_2$, the cycles that are the images of $\hat{x}_1$ and $\hat{x}_2$ in $\tilde{Y}$. Then $\tilde{Y}\langle S_1, S_2\rangle$ is a minimal $\tilde{R}$-resolution of k.

To prove that $\tilde{R}\to R/I$ is Golod, it suffices to verify formula (3.2). Since a_1, a_2 is a regular sequence in m^2, we have $P_R(P_{\tilde{R}})^{-1}=(1-z^2)^2$; using Theorem 2.2 for $P_R(P_{R/I})^{-1}$, we see that it suffices to prove

$$P_{\tilde{R}}^{R/I}=1+(n-2)z+(2n-5)z^2+(n-3)\sum_{i=3}^{\infty} iz^i \tag{3.7}$$

where $n=\mu(I)$.

The Poincaré series $P_{\tilde{R}}^{R/I}$ is obtained by studying the homology H of $\tilde{Y}\langle S_1,S_2\rangle\otimes_{\tilde{R}}(R/I)=Y\langle S_1,S_2\rangle$. It is clear that $\{1\}$, $\{x_j \mid 3\le j\le n\}$ and $\{y_j \mid 4\le j\le n-1\}\bigcup\{y_j' \mid 1\le j\le n-1\}$ represent bases of H_i for $i=0,1,2$.

Fix an integer $k\ge 1$. We claim that there are $(n-3)(2k+1)$ cycles of form (3.9a) and $(n-3)(2k)$ cycles of form (3.9b) that represent bases for H_{2k+1} and H_{2k} respectively; from this (3.7) follows immediately. (Here $+\cdots$ denotes terms of lower total order in S_1 and S_2.)

$$\begin{cases} x_l S_1^{(a)}S_2^{(k-a)}+\cdots, & 0\le a\le k, \quad 4\le l\le n,\\ z_l S_1^{(a)}S_2^{(k-1-a)}+\cdots, & 0\le a\le k-1, \quad 4\le l\le n,\end{cases} \tag{3.9a}$$

$$\begin{cases} y_1 S_1^{(a)}S_2^{(k-1-a)}+\cdots, & 1\le a\le k-1,\\ y_l S_1^{(a)}S_2^{(k-1-a)}+\cdots, & 0\le a\le k-1, \quad 4\le l\le n-1,\\ y_l' S_1^{(a)}S_2^{(k-1-a)}+\cdots, & 0\le a\le k-1, \quad 4\le l\le n-1,\\ y_1' S_1^{(a)}S_2^{(k-1-a)}+y_2' S_1^{(a+1)}S_2^{(k-2-a)}+\cdots, & 0\le a\le k-2,\\ y_1' S_1^{(k-1)}+\cdots, & \\ y_2' S_2^{(k-1)}+\cdots. & \end{cases} \tag{3.9b}$$

It is not difficult to see that such cycles exist. To see that they generate H_i for $i\ge 3$ we induct on the polynomial degree in S_1 and S_2. Let $\eta=\sum_{a=0}^{k}\eta_a$ be a cycle in $Y\langle S_1,S_2\rangle$ with each η_a a (divided power) polynomial of degree a in S_1 and S_2 with coefficients in Y. In particular in the highest degree

$$\eta_k=\sum_{i+j=k}\zeta_{ij}S_1^{(i)}S_2^{(j)}$$

where each ζ_{ij} must be a cycle in Y. Recall that $d(z_1S_1^{(i+1)}S_2^{(j)})=wS_1^{(i)}S_2^{(j)}$ (since $Y_5=0$ boundary elements in Y_4 are all zero). We can choose χ, an R-linear combination of cycles from (3.9a) and (3.9b), and an element σ of

$$(1,x_1,x_3,y_1',y_3',z_1)\cdot(S_1,S_2)^{(k+1)}$$

so that the leading term of $\eta-(\chi+d\sigma)$ is

$$\sum_{a=0}^{k-1}(u_a x_1+u'_a y'_1+u''_a z_2)S_1^{(a)}S_2^{(k-a)}$$
$$+\sum_{a=0}^{k}(v_a+v'_a x_3+v''_a y'_3+d\varrho_a)S_1^{(a)}S_2^{(k-a)}$$

for some ϱ_a in Y and $u_a,u'_a,u''_a,v_a,v'_a,v''_a$ in R. Applying d to the cycle $\eta-\chi-d(\sigma+\sum_{a=0}^{k}\varrho_a S_1^{(a)}S_2^{(k-a)})$, and collecting coefficients of each $S_1^{(a)}S_2^{(k-a)}$, we see that $\bar{u}_a=\bar{u}'_a=\bar{u}''_a=\bar{v}_a=\bar{v}'_a=\bar{v}''_a=0$ for all a. Hence the cycle $\eta-\chi$ differs by a boundary from a cycle that is a polynomial of degree less than k in S_1 and S_2; by induction this cycle differs by a boundary from a linear combination of cycles of (3.9a) and (3.9b).

We conclude the argument by showing that the cycles of (3.9a) and (3.9b) represent linearly independent classes in homology. Suppose that some R-linear combination of them is a boundary. To show that all the coefficients are zero, we make the following observation (also to be used in Case (D)). If s_{pq} and ζ are homogeneous elements in Y so that $\sum s_{pq}S_1^{(p)}S_2^{(q)}$ is a homogeneous element in $Y\langle S_1,S_2\rangle$ and

$$d(\sum s_{pq}S_1^{(p)}S_2^{(q)})=\zeta S_1^{(i)}S_2^{(j)}+\cdots$$

where the other terms have degree at most $i+j$ in S_1 and S_2, then

$$\begin{aligned}\zeta&=ds_{ij}+(-1)^{\deg s_{ij}}(s_{i+1\,j}x_1+s_{i\,j+1}x_2),\\ 0&=ds_{i+1\,j}+(-1)^{\deg s_{ij}}(s_{i+2\,j}x_1+s_{i+1\,j+1}x_2),\\ 0&=ds_{i\,j+1}+(-1)^{\deg s_{ij}}(s_{i+1\,j+1}x_1+s_{i\,j+2}x_2).\end{aligned}\tag{3.10}$$

It suffices to handle three cases:

(a) Suppose $(\sum_{l=4}^{n}\alpha_l x_l)S_1^{(i)}S_2^{(j)}+\cdots$ is a boundary. In the notation of (3.10), $s_{cd}=0$ if $c+d=i+j+2$ by degree considerations, and similarly s_{cd} is in R if $c+d=i+j+1$. Thus by (3.10) we see that $\zeta=\sum\alpha_l x_l$ is a boundary in $Y\langle S_1,S_2\rangle$, which implies $\bar{\alpha}_l=0$ for $l=4,\dots,n$.

(b) Suppose $(\sum_{l=4}^{n}\alpha_l z_l)S_1^{(i)}S_2^{(j)}+\cdots$ is a boundary. Now

$$\deg s_{cd}=\begin{cases}4 & \text{if } c+d=i+j,\\ 2 & \text{if } c+d=i+j+1,\\ 0 & \text{if } c+d=i+j+2.\end{cases}$$

Since the cycles x_1 and x_2 represent the beginning of a basis of $H_1(Y)$, it follows that s_{cd} is in m for $c+d=i+j+2$ and hence there exist t_{cd} in Y_1 such that $dt_{cd}=s_{cd}$. Of course $(dt_{cd})x_l=d(t_{cd}x_l)$ for all l. By (3.10) both $s_{i+1\,j}+t_{i+2\,j}x_1+t_{i+1\,j+1}x_2$ and $s_{i\,j+1}+t_{i+1\,j+1}x_1+t_{i\,j+2}x_2$ are cycles; thus they are in the R-submodule of Y_2 spanned by dY_3 and $\{y_1,\dots,y_{n-1},y'_1,\dots,y'_{n-1}\}$. Consequently there exist R-linear combinations η_1 and η_2 of the y_i and y'_i so that $\zeta=\sum_{l=4}^{n}\alpha_l z_l$ equals

$$\begin{aligned}&(\eta_1-t_{i+2\,j}x_1-t_{i+1\,j+1}x_2)x_1+(\eta_2-t_{i+1\,j+1}x_1-t_{i\,j+2}x_2)x_2+\text{boundary}\\ &=\eta_1x_1+\eta_2x_2+\text{boundary}.\end{aligned}$$

Recall that $x_i^2=0$ and $x_1x_2=-x_2x_1$ since Y is a DG-algebra, and each $\eta_i x_i$ is in the R-submodule of Y_3 spanned by dY_4 and $\{z_1,z_2,z_3\}$. It follows that $\bar{\alpha}_l=0$ for $l=4,\dots,n$.

(c) Label the leading terms of the cycles of (3.9b) as follows:

$$\xi_1=y_1S_1^{(1)}S_2^{(k-2)},\qquad \xi_2=y_1S_1^{(2)}S_2^{(k-3)},\dots,\xi_{k-1}=y_1S_1^{(k-1)},$$

and ξ_k to $\xi_{2k(n-3)}$ the other leading terms in some order. Suppose $\sum_{l=1}^{2k(n-3)}\alpha_l\xi_l+\cdots$ is a boundary. Fix i and j with $i+j=k-1$. By (3.10) and degree considerations $s_{cd}=0$ if $c+d=k+1$ and s_{cd} is a 1-cycle if $c+d=k$. The coefficient ζ of $S_1^{(i)}S_2^{(j)}$ is equal to $ds_{ij}-s_{i+1\,j}x_1-s_{i\,j+1}x_2$. The product of two 1-cycles is in the R-submodule of Y_2 spanned by dY_3 and $\{y_1,y_2,y_3\}$; hence so is ζ. Since $\{y_1,\dots,y_{n-1},y_1',\dots,y_{n-1}'\}$ represents a basis for $H_2(Y)$ it follows that $\bar{\alpha}_l=0$ for $k\le l\le 2k(n-3)$. We are left with a boundary of form $\sum_{l=1}^{k-1}\alpha_l y_1S_1^{(a)}S_2^{(k-1-a)}+$ l.o.t.(S_1,S_2). By (3.10) there is a system of equations

$$\begin{aligned}
0&=ds_{0\,k-1}-s_{1\,k-1}x_1-s_{0\,k}x_2,\\
\alpha_1y_1&=ds_{1\,k-2}-s_{2\,k-2}x_1-s_{1\,k-1}x_2,\\
\alpha_2y_1&=ds_{2\,k-3}-s_{3\,k-3}x_1-s_{2\,k-2}x_2,\\
&\vdots\\
\alpha_{k-1}y_1&=ds_{k-1\,0}-s_{k\,0}x_1-s_{k-1\,1}x_2,
\end{aligned}$$

and each s_{cd} is a 1-cycle if $c+d=k$. Thus $s_{cd}=\sum_{i=1}^n u_i(c,d)x_i+d\sigma_{cd}$ for some $u_i(c,d)\in R$ and $\sigma_{cd}\in Y_2$ if $c+d=k$. All terms in the product $s_{cd}x_1$ or $s_{cd}x_2$ are boundaries in Y, except possibly those involving x_1x_2, x_1x_3, or x_2x_3, which are equivalent to the independent cycles y_3, $-y_2$, and y_1 respectively. The top equation yields $\bar{u}_3(1,k-1)=0$; the next then yields $\bar{\alpha}_1=\bar{u}_3(1,k-1)=0$ and $\bar{u}_3(2,k-2)=0$. Proceeding inductively we obtain $\bar{\alpha}_l=\bar{u}_3(l,k-l)=0$ and $\bar{u}_3(l+1,k-l-1)=0$ for each l.

Case (D). We shall prove that there is a regular sequence a_0,a_1 in I so that the natural map $R/(a_0,a_1)\to R/I$ is Golod. Both $\hat{Y}$ and Y are defined as above. For convenience we reindex the basis elements of the reduced homology of Y as follows: they are the equivalence classes of cycles $x_0,\dots,x_{n-1}$ in degree 1; $y_1,\dots,y_{n-1},y_1',\dots,y_{n-1}'$ in degree 2; $z_0,\dots,z_{n-1}$ in degree 3, and w in degree 4. The multiplication on Y is given by $x_iz_j=\delta_{ij}w$, $y_iy_j'=\delta_{ij}w$, and with the exception of the products listed below all other products are boundaries:

$$\begin{aligned}
x_0x_i&=y_i+b,\\
x_iy_i'&=z_0+b,\qquad 1\le i\le p\\
x_0y_i'&=-z_i+b.
\end{aligned}$$

For each $i=0,1$, let $\hat{x}_i$ be a preimage of x_i in $\hat{Y}$, with $dx_i=a_i$, chosen so that a_0,a_1 is a regular sequence in I. Let $\tilde{R}=R/(a_0,a_1)$, and adjoin variables S_0 and S_1 of degree two to kill the images of $\hat{x}_i$ in $\tilde{Y}=\hat{Y}\otimes_R\tilde{R}$. Once again, to show that $\tilde{R}\to R/I$

is Golod it suffices to verify (3.2), or equivalently in this case,

$$P_{\tilde{R}}^{R/I} = 1+(n-2)z+(2n-p-2)z^2+\sum_{i=3}^{\infty}[i(n-p-1)+p-1]z^i \tag{3.11}$$

where $n=\mu(I)$ and $p=\dim \Lambda_1^2$.

The Poincaré series $P_{\tilde{R}}^{R/I}$ is obtained by studying the homology H of $\tilde{Y}\langle S_0,S_1\rangle\otimes_{\tilde{R}}(R/I)=Y\langle S_0,S_1\rangle$. It is clear that $\{1\},\{x_j \mid 2\le j\le n-1\}$, and $\{y_j \mid p+1\le j\le n-1\}\cup\{y_j' \mid 1\le j\le n-1\}$ represent bases of H_i for $i=0,1,2$. Fix an integer $k\ge 1$. We claim that there are $p-1+(2k+1)(n-p-1)$ cycles of form (3.12a) and $p-1+(2k)(n-p-1)$ cycles of form (3.12b) that represent bases for H_{2k+1} and H_{2k}; from this (3.11) follows immediately.

$$\begin{cases} x_lS_1^{(k)}+\cdots, & 2\le l\le p,\\ x_lS_0^{(a)}S_1^{(k-a)}+\cdots, & 0\le a\le k, \quad p+1\le l\le n-1,\\ z_lS_0^{(a)}S_1^{(k-1-a)}+\cdots, & 0\le a\le k-1, \quad p+1\le l\le n-1,\end{cases} \tag{3.12a}$$

$$\begin{cases} y_lS_0^{(a)}S_1^{(k-1-a)}+\cdots, & 0\le a\le k-1, \quad p+1\le l\le n-1,\\ y_l'S_0^{(a)}S_1^{(k-1-a)}+\cdots, & 0\le a\le k-1, \quad p+1\le l\le n-1,\\ y_l'S_1^{(k-1)}+\cdots, & 2\le l\le p.\end{cases} \tag{3.12b}$$

It is not difficult to see that such cycles exist, and to see that they generate H_i for $i\ge 3$ we shall induct on the polynomial degree in S_0 and S_1. If η is a cycle in $Y\langle S_0,S_1\rangle$ with leading term of degree k in S_0 and S_1, then we can find χ, an R-linear combination of cycles from (3.12a) and (3.12b), and an element σ of

$$(1,x_1,\ldots,x_p y_1',\ldots,y_p',z_0)(S_0,S_1)^{(k+1)}$$

so that the leading term of $\eta-(\chi+d\sigma)$ is

$$(u_0y_1'+d\varrho_0)S_1^{(k)}+\sum_{a=1}^{k}\left(\sum_{l=1}^{p}[u_{la}x_l+u_{la}'y_l']+u_a+v_az_0+d\varrho_a\right)S_0^{(a)}S_1^{(k-a)}$$

with u_a,u_{la},u_{la}',v_a in R and ϱ_a in Y. As in case (C) we apply d to see that $\bar{u}_a=\bar{u}_{la}=\bar{u}_{la}'=\bar{v}_\alpha=0$ for all l and a, and the argument concludes exactly as in case (C). To verify that the cycles of (3.12) represent linearly independent classes in homology we again adopt the notation of (3.10) and consider the three cases, which are sufficient.

(a) If $(\sum_{l=2}^{n-1}\alpha_lx_l)S_0^{(i)}S_1^{(j)}+\cdots$ is a boundary, then $\sum_{l=2}^{n-1}\alpha_lx_l$ is a boundary in $Y\langle S_0,S_1\rangle$. Since $\{x_2,\ldots,x_{n-1}\}$ represents a basis for H_1, it follows that each $\bar{\alpha}_l$ is zero.

(b) If $(\sum_{l=p+1}^{n-1}\alpha_lz_l)S_0^{(i)}S_1^{(j)}+\cdots$ is a boundary, then $\zeta=\sum\alpha_lz_l$ has the form $ds_{ij}+s_{i+1\,j}x_0+s_{i\,j+1}x_1$, where $\deg s_{i+1\,j}=\deg s_{i\,j+1}=2$. Hence ζ is in the R-submodule of Y_3 spanned by dY_4 and $\{z_0,\ldots,z_p\}$, and therefore each $\bar{\alpha}_l$ is zero.

(c) If $\zeta S_0^{(i)}S_1^{(j)}+\cdots$ is a boundary with $\zeta=\sum_{l=p+1}^{n-1}\alpha_ly_l+\sum_{l=2}^{n-1}\alpha_l'y_l'$, then $\zeta=ds_{ij}-s_{i+1\,j}x_0-s_{i\,j+1}x_1$, where $\deg s_{i+1\,j}=\deg s_{i\,j+1}=1$. Hence ζ is in the R-submodule of Y_2 spanned by dY_3 and $\{y_1,\ldots,y_p\}$, and therefore each $\bar{\alpha}_l$ and $\bar{\alpha}_l'$ is zero. □

Let R be a regular local ring over a field k and let I be a grade four Gorenstein ideal in R. Then $S=R/I$ is either a complete intersection or it is a Gorenstein local ring of codimension three or four. If S is a complete intersection, most of its homological properties are known. The homotopy Lie algebra $\pi^*(S)$ is finite-dimensional (since $\pi^{\geq 3}(S)=0$), the Ext-algebra is Noetherian; i.e. the λ-dimension of $\mathrm{Ext}_S(k,k)$ is zero (Roos [23]); also from [23] the finistic global dimension of $\mathrm{Ext}_S(k,k)$ is equal to $e_2(S)$, which is at most four, and the Poincaré series $P_S^M(z)$ is rational for every finitely generated module M.

Theorem 3.1 provides much information for the case in which S is not a complete intersection.

Corollary 3.2. *Let R be a regular local ring over a field k not of characteristic two. Let I be a grade four Gorenstein ideal in R such that $S=R/I$ is not a complete intersection. Then*

(i) *$P_S^M(z)$ is rational for every finitely generated S-module M.*

(ii) *There is an exact sequence of graded Lie algebras*

$$0\to L(W)\to\pi^*(S)\to\pi^*(\tilde{R})\to 0$$

where $L(W)$ is a free Lie algebra, $\dim_k \pi^1(\tilde{R})=\dim_k \pi^1(S)=e_1(S)$, $\dim_k \pi^2(\tilde{R})=1$ *or* 2, *and* $\dim_k \pi^i(\tilde{R})=0$ for $i\geq 3$.

(iii) *$e_i(S)>0$ for all $i\geq 1$ and $e_{2i}(S)>e_i(S)$ for all odd $i>1$.*

(iv) *The finistic global dimension of $\mathrm{Ext}_S(k,k)$ is at most three.*

(v) *The λ-dimension of $\mathrm{Ext}_S(k,k)$ is one; i.e. the Ext-algebra is coherent.*

Proof. Property (i) follows from a result of Levin (unpublished): if $\tilde{R}$ is a complete intersection and $\tilde{R}\twoheadrightarrow S$ is a Golod map, then $P_S^M(z)$ is rational for every finitely generated module M. The Lie algebra definition (3.1) of a Golod map gives us (ii), and (iii) follows from Jacobsson [13]. The last two statements follow from Roos [23]. □

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Note added in proof

In Theorem 2.2 and elsewhere, by inflation of R (see N. Bourbaki, Algèbre Commutative, Chap. IX, 38–39) we may pass to a flat extension, for which the residue field k has $\sqrt{2}$ and $\sqrt{-1}$, the only roots actually required. Since the Poincaré series remains unchanged, we need only assume char $k \neq 2$.

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